Econ 802
Answers to First Midterm

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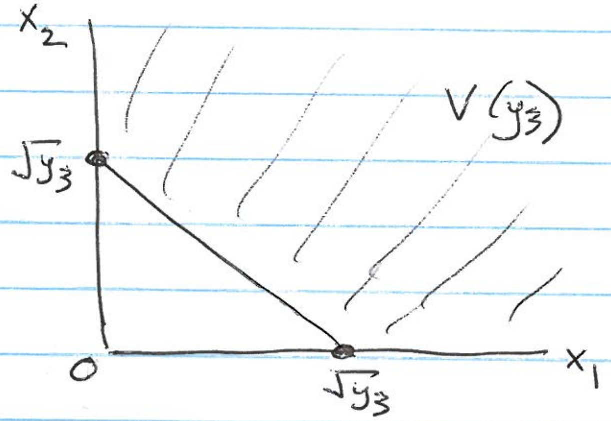
1. (a) First define $x_1 = -y_1 \geq 0$ and $x_2 = -y_2 \geq 0$. The set $V(y_3)$ is the set of (x_1, x_2) vectors such that $(x_1+x_2)^2 \geq y_3$ or $x_1 + x_2 \geq \sqrt{y_3}$. This gives:



Non-empty? Yes, there is always a way to produce y_3.

Monotonic? Yes, if $x \in V(y_3)$ then $x' \in V(y_3)$ where $x' \geq x$.

Closed? Yes, $V(y_3)$ contains all of its boundary points.

Bounded? No. For any y_3 there are points arbitrarily far from the origin that are in $V(y_3)$.

Convex? Yes, for any pair of points $x \in V(y_3)$ and $x' \in V(y_3)$, the line segment between them is in the set.

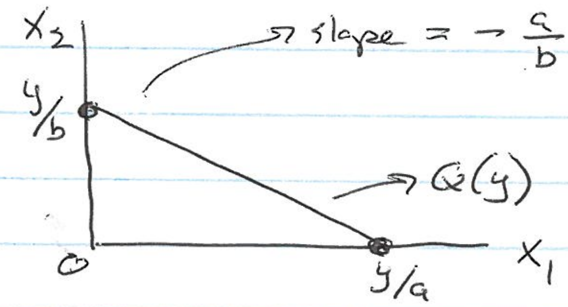
Strictly convex? No. It is possible to choose a pair of points in $V(y_3)$ such that all points on the line segment between them are on the boundary of $V(y_3)$, not in the interior.

(b) Yes. Because $V(y_3)$ is non-empty, there is some $x \in V(y_3)$. This point is on some downward-sloping isocost line. All points on higher isocost lines can be ignored (they cannot be solutions). The set of points $(x_1, x_2) \geq 0$ on or below this isocost line is compact (closed and bounded). The same is true for the intersection of this set with $V(y_3)$. From the Weierstrasse Theorem, a continuous function must reach a minimum value somewhere on a compact set. Therefore the cost min problem has a solution.

Graphically, it is clear that the solution is either at the corner $x_1 = \sqrt{y_3}$ and $x_2 = 0$; or the corner $x_1 = 0$, and $x_2 = \sqrt{y_3}$; or along the line segment connecting these points.

(c) No. Use the production function $y_3 = f(x) = (x_1 + x_2)^2$. Write profit as $p(x_1+x_2)^2 - w_1 x_1 - w_2 x_2$. Even if we don't use input x_2 at all (set it equal to zero), we can have unbounded profit. $p x_1^2 - w_1 x_1$ is a quadratic function of x_1 and we can make it arbitrarily large by choosing a sufficiently large x_1.

2\. (a) (i) The isoquants are straight lines:



The isocost lines have slope $= -\frac{w_1}{w_2}$. If the isocost lines are steeper $\left(\frac{w_1}{w_2} > \frac{a}{b}\right)$ we have the corner solution $x_1 = 0$, $x_2 = \frac{y}{b}$

If the isocost lines are flatter $\left(\frac{w_1}{w_2} < \frac{a}{b}\right)$ we have the corner solution $x_1 = y/a$, $x_2 = 0$.

If the isocost lines have the same slope as the isoquant $\left(\frac{w_1}{w_2} = \frac{a}{b}\right)$ then every point along the isoquant is a solution.

2 (a) (ii). Use the Lagrangean $w_1x_1 + w_2x_2 - \lambda\left[x_1^a x_2^b - y\right]$

FOC: $w_1 - \lambda a x_1^{a-1} x_2^b = 0$, $w_2 - \lambda x_1^a b x_2^{b-1} = 0$, $x_1^a x_2^b - y = 0$; divide the first by the second $\Rightarrow$ $\frac{w_1}{w_2} = \frac{a}{b}\frac{x_2}{x_1}$

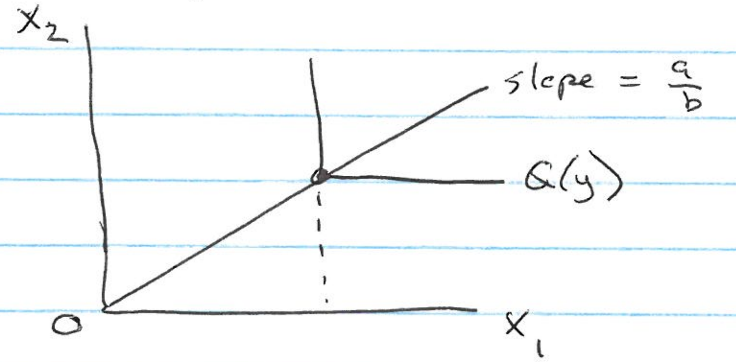
So $x_2 = \frac{x_1 w_1}{w_2}\frac{b}{a}$; substitute into the last equation to get

$$x_1^a\left[\frac{x_1 w_1}{w_2}\frac{b}{a}\right]^b = y$$

$$\Rightarrow x_1^{a+b} = y\left[\frac{w_2}{w_1}\frac{a}{b}\right]^b \Rightarrow x_1 = y^{\frac{1}{a+b}}\left[\frac{w_2 a}{w_1 b}\right]^{\frac{b}{a+b}}$$

and symmetrically, $x_2 = y^{\frac{1}{a+b}}\left[\frac{w_1 b}{w_2 a}\right]^{\frac{a}{a+b}}$

(iii) The isoquants have corners along the ray with slope $\frac{a}{b}$:



No matter what the input prices are, we get $x_1 = \frac{y}{a}$ and $x_2 = \frac{y}{b}$.

(Set $ax_1 = bx_2 = y$ to get the cost-minimizing way to produce y.)

(b) The student is likely to get the right answer for the Cobb-Douglas case (ii). The production function is differentiable. It is also strictly quasi-concave, so the FOC are satisfied at a unique point and this point is the solution. It is also an interior point, so the non-negativity constraints $x_1 \geq 0$, $x_2 \geq 0$ are irrelevant.

The student may get the wrong answer in the linear case (i). Although the production function is differentiable, due to linearity, x_1 and x_2 drop out of the FOC

In general the solution is at a corner where one of the non-negativity constraints is binding, so doing the FOC correctly requires Kuhn-Tucker multipliers. The student might also have trouble with the existence of multiple solutions when $\frac{w_1}{w_2} = \frac{a}{b}$. And finally, the student might be puzzled about the SOC because all elements of the Hessian $\frac{\partial^2 f}{\partial x^2}$ are zero. The student may also get the wrong answer in the Leontief case (iii) because the production function is not differentiable at the cost-minimizing point.

(c) This is certainly true for case (iii) because the input demands do not change when the wage goes up, so there is only a price effect. It is not true for case (ii) because with a Cobb-Douglas production function the cost shares are fixed (they are given by the exponents in the production functions). It could be true in case (i) if we are at a corner solution and the price change is small enough. But if we are at an interior solution, even a tiny wage increase will lead to no labor demand at all. Mathematically, let labor be input 1. We are interested in the share $\frac{w_1 x_1}{w_1 x_1 + w_2 x_2} = \frac{1}{1 + 1/s}$ where $s = \frac{w_1 x_1}{w_2 x_2} \equiv \hat{w}\hat{x}$.

Labor's share goes up when $\hat{w} = \frac{w_1}{w_2}$ increases if the following derivative is positive:

$$\frac{\partial s}{\partial \hat{w}} = \hat{x} + \hat{w}\frac{\partial \hat{x}}{\partial \hat{w}} = \hat{x}\left[1 - \sigma\right] \text{ where } \sigma = \text{elasticity of substitution}$$

elastic $(\sigma > 1) \Rightarrow$ labor's share $\downarrow$; inelastic $(\sigma < 1) \Rightarrow$ labor share $\uparrow$

3 (a) Let $p' \geq p^0$. We want to show that $\pi(p') \geq \pi(p^0)$. Let y' be optimal for p' and let y^0 be optimal for p^0. Then $\underbrace{\pi(p') = p'y' \geq p'y^0}_{\text{follows from the fact that } y' \text{ is optimal at } p'} \underbrace{\geq p^0y^0}_{\text{follows from } p' \geq p^0} = \pi(p_0)$

Intuition: when some prices rise, the firm is always at least as well off as it was before if it keeps doing the same thing (y^0). If in addition the firm re-optimizes (to y') then it might do even better.

(b) Let y^0 be optimal for p^0. This implies that $p^0y^0 \geq p^0y$ for all $y \in Y$. Therefore $tp^0y^0 \geq tp^0y$ for all $y \in Y$, where $t > 0$ is a scalar. This implies that y^0 is optimal at the price vector tp^0. Thus

$\pi(tp^0) = tp^0y^0 = t\pi(p^0)$ which is what we need.

Intuition: if we multiply the prices by t, we are multiplying the objective function by t. This doesn't change behavior.

(c) We want to show $\pi[tp + (1-t)p'] \leq t\pi(p) + (1-t)\pi(p')$ for $0 \leq t \leq 1$. Define $p'' = tp + (1-t)p'$ and let y'' be optimal for p''. Then $\pi[tp + (1-t)p'] = \pi(p'') = p''y''$ $= tpy'' + (1-t)p'y'' \leq t\pi(p) + (1-t)\pi(p')$ because py'' cannot be larger than $\pi(p)$ and $p'y''$ cannot be larger than $\pi(p')$.

Intuition: The firm can always respond to price changes passively by not changing behavior. This gives a linear expression for profit. If the firm instead optimizes, profit may be above or on this line, but cannot be below it.

4 (a) Due to CRS we can write the cost function in the form $c(w,y) = y\,c(w,1)$. Then use Shepherd's Lemma: $x_i(w,y) = \frac{\partial c(w,y)}{\partial w_i} = y\frac{\partial c(w,1)}{\partial w_i}$

so we can set

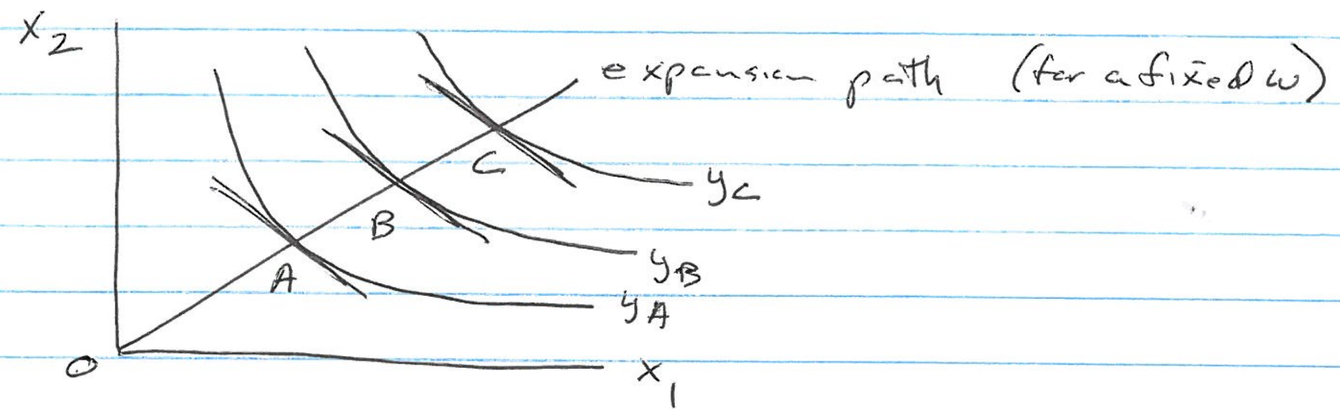
$z_i(w) = \frac{\partial c(w,1)}{\partial w_i}$ for all $i = 1 \ldots n$.

(b) From part (a) we have $x_1(w,y) = y\,z_1(w)$

$x_2(w,y) = y\,z_2(w)$

and therefore $\frac{x_1(w,y)}{x_2(w,y)} = \frac{z_1(w)}{z_2(w)}$.

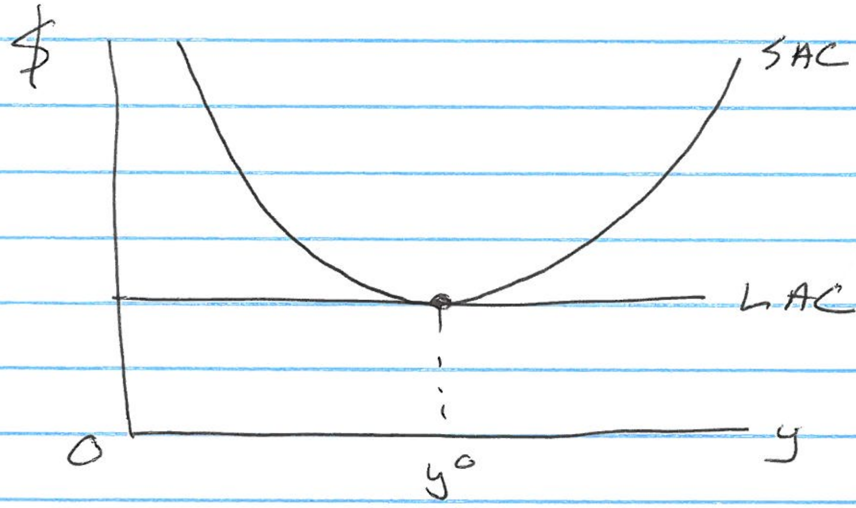
Notice that the right hand side does not depend on y. This shows that for all levels of y, the input ratio that minimizes cost must be the same. Therefore the expansion path must be a ray from the origin:



The rate of technical substitution (or technical rate of substitution) is the slope of the isoquant through a point. Because the isocost lines for a fixed w must all have the same slope, and cost min implies that the slopes of the isocost lines are equal to the slopes of the isoquants, it must be true that TRS is constant as we move out along a ray from the origin, holding the input ratio constant.

(c) We know from CRS that the long run AC curve is horizontal: $c(w, y) = yc(w, 1)$ implies $LAC = \frac{c(w,y)}{y} = c(w,1)$ which does not depend on y.

The short run average cost curve SAC is generally above LAC because the firm generally cannot use the inputs (x_1, x_2) that would be optimal in the long run. But we know that short run cost $c(y, x_2)$ is equal to long run cost $c(y)$ if $x_2 = x_2(y)$ happens to be the long run optimal level of x_2. So we expect the cost curves to look like this:



Note: This is SAC for a specific fixed value of x_2; if we had a different level of x_2 we would have a different SAC.

Let x_2^0 be the optimal demand for x_2 in the long run at y^0.

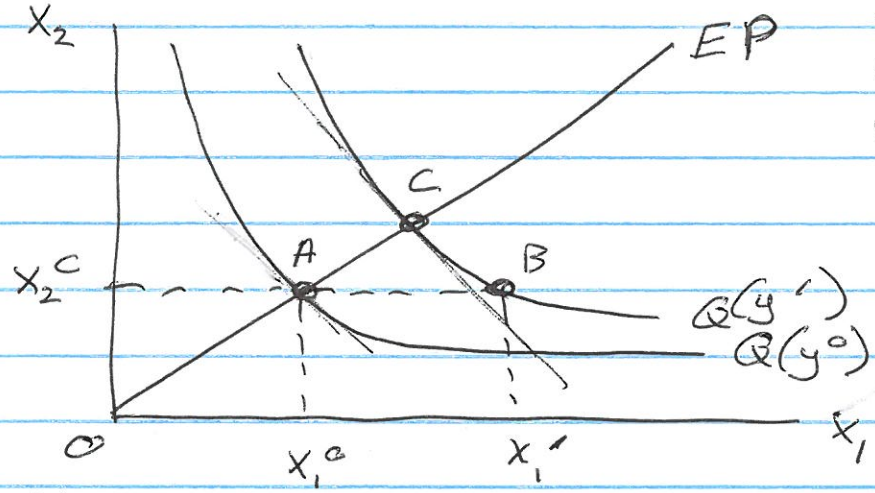
Then $\underbrace{c(y^0)}_{\text{long run total cost}} = \underbrace{c(y^0, x_2^0)}_{\text{short run total cost}} \Rightarrow \underbrace{\frac{c(y^0)}{y^0}}_{LAC(y^0)} = \underbrace{\frac{c(y^0, x_2^0)}{y^0}}_{SAC(y^0)}$

So the curves touch at the output y^0. Furthermore, at any other output $y' \neq y^0$, we must have $c(y') \not< c(y', x_2^0)$ if x_2^0 is not optimal in the long run for the output y'.

If this is true then $\underbrace{\frac{c(y')}{y'}}_{LAC(y')} < \underbrace{\frac{c(y', x_2^0)}{y'}}_{SAC(y')}$ for all $y' \neq y^0$

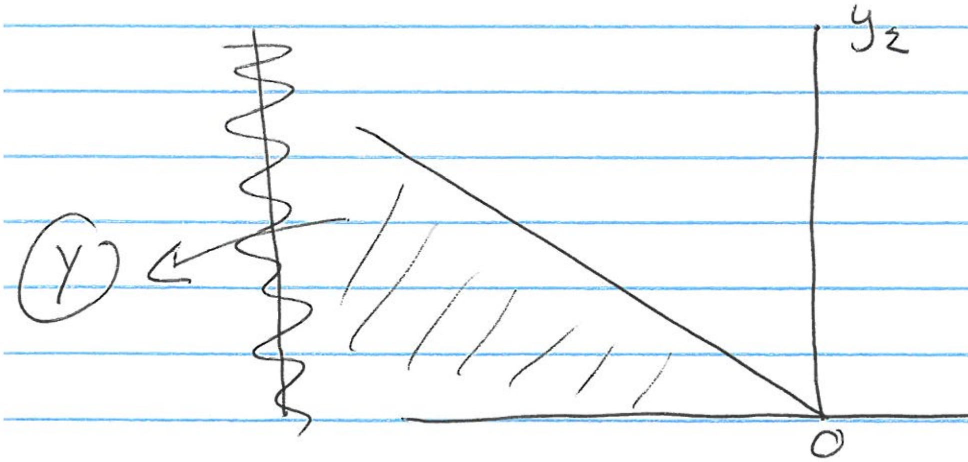
So the curves can only touch at the output y^0.

To relate this to part (b), use the fact that the expansion path is linear:



In the long run, at y^0 the firm uses (x_1^0, x_2^0). (point A) However at some other output y', if $x_2 = x_2^0$ then in the short run the firm is forced off the expansion path to a point like B. This implies higher costs in the short run than what it could achieve in the long run by going to point C (back on the expansion path).

5(a) In general, this statement is false for two reasons. First, we could have a convex Y without having a solution for profit max. Suppose Y looks like this:



If the isoprofit lines are <u>flatter</u> than the slope of the upper boundary for Y, then there is no upper bound on profit.

Note that Y is convex but not strictly convex.

Second, it is possible for the same Y to have a solution that is unique. If the isoprofit lines are steeper than the upper boundary of Y, the unique solution is $y_1 = y_2 = 0$, and max profit is zero.

(b) Suppose we observe price vectors p^t and production plans y^t for periods $t = 1 \ldots T$. WAPM says that

$$p^t y^t \geq p^t y^s \text{ for all } t \text{ and } s.$$

For a particular t and s, we have $p^t(y^t - y^s) \geq 0$

and $-p^s(y^t - y^s) \geq 0$

Summing these inequalities gives $(p^t - p^s)(y^t - y^s) \geq 0$ or $\Delta p \cdot \Delta y \geq 0$. This shows that if an output price rises, the firm cannot produce less of that output, and if an input price rises, the firm cannot use more of that input.

(c) The FOC is $p\dfrac{\partial f(x)}{\partial x} = w$. Let the solution be $x(p, w)$ (assuming a solution exists). This gives the identity

$$p\frac{\partial f[x(w)]}{\partial x} \equiv w$$

where we drop p as an argument of $x(p, w)$ because it is held constant.

Differentiate both sides with respect to w to get

$$p\frac{\partial^2 f[x(w)]}{\partial x^2} \cdot \frac{\partial x(w)}{\partial w} = I.$$

Because the Hessian is negative definite, it is non-singular and we have $\dfrac{\partial x(w)}{\partial w} = \left[p\dfrac{\partial^2 f[x(w)]}{\partial x^2}\right]^{-1}$.

① The Hessian is symmetric so this is also true for $\dfrac{\partial x}{\partial w}$.

② The Hessian is negative definite so this is also true for $\dfrac{\partial x}{\partial w}$. In particular, the input demand curves slope down due to $\dfrac{\partial x_i}{\partial w_i} < 0$ for all $i = 1 \ldots n$.